Market Failure: Public Goods

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Consider an economy in which there is a public good (x) and a private good (y). The economy is described by a collection

$$[(u_1, \bar{y}_1), ..., (u_n, \bar{y}_n), c],$$

where $\bar{y}_i \in \mathbb{R}_+$ and

$$u_i : \mathbb{R}^2_+ \to \mathbb{R}$$

identify the preferences an initial endowment of private good of individual $i \in \{1, ..., n\}$, and

$$c : \mathbb{R}_+ \to \mathbb{R}_+,$$

identifies the cost (in units of the private good) of producing public good. Initially, the economy lacks any public good.

An *allocation* is a vector $(x, y) = (x, y_1, ..., y_n) \in \mathbb{R}_+ \times \mathbb{R}_+^n$. An allocation (x, y) is *feasible* if

$$\sum_{i=1}^{n} y_i + c(x) \leq \sum_{i=1}^{n} \bar{y}_i.$$

Allocation (x', y') is *Pareto superior* to (x, y) if

$$u_i(x', y_i') \geq u_i(x, y_i)$$

holds for all $i \in \{1, ..., n\}$ and

$$\sum_{i=1}^{n} u_i(x', y_i') > \sum_{i=1}^{n} u_i(x, y_i).$$

An allocation is *Pareto optimal* if it is feasible and there is no feasible Pareto superior allocation.

For every profile of *weights* $\lambda = (\lambda_1, ..., \lambda_n) \in \triangle^n$, define

$$P(\lambda) := \quad \begin{array}{l} \max\limits_{(x,y)\in\mathbb{R}_+\times\mathbb{R}_+^n} \sum_{i=1}^n \lambda_i u_i(x, y_i) \\ s.t. \ \sum_{i=1}^n y_i + c(x) \leq \sum_{i=1}^n \bar{y}_i. \end{array}$$

Proposition. An allocation (x, y) is Pareto optimal if and only if it solves $P(\lambda)$ for some $\lambda \in \triangle^n$.

Proof.

$\Rightarrow$) Let $(x, y_1, ..., y_n)$ be a solution to $P(\lambda)$ for some $\lambda \gg 0$. If there is a feasible Pareto superior allocation $(x', y_1', ..., y_n')$, then

$$\sum_{i=1}^{n} \lambda_i u_i(x', y_i') > \sum_{i=1}^{n} \lambda_i u_i(x, y_i),$$

which contradicts that $(x, y_1, ..., y_n)$ is a solution to $P(\lambda)$.

$\Leftarrow$) More involved.

Assume that each u_i is increasing in x and y, differentiable and concave, and that c is differentiable, increasing and convex. Then for all $\lambda \in \triangle^n$, $P(\lambda)$ is a convex problem, and its solutions are critical points of the Lagrangian:

$$\mathcal{L}(x, y_1, ..., y_n, \mu) = \sum_{i=1}^{n} \lambda_i u_i(x, y_i) + \mu \left(\sum_{i=1}^{n} \bar{y}_i - \sum_{i=1}^{n} y_i - c(x) \right).$$

These critical points are solutions to the system of equations:

$$(x) \qquad \frac{\partial \mathcal{L}}{\partial x} = \sum_{i=1}^{n} \lambda_i \frac{\partial u_i}{\partial x} - \mu c'(x) = 0$$

$$(y_i) \qquad \frac{\partial \mathcal{L}}{\partial y_i} = \lambda_i \frac{\partial u_i}{\partial y_i} - \mu = 0, \ \forall i$$

$$(\mu) \qquad \frac{\partial \mathcal{L}}{\partial \mu} = \sum_{i=1}^{n} \bar{y}_i - \sum_{i=1}^{n} y_i - c(x) = 0.$$

By equation (y_i) we get

$$\lambda_i \frac{\partial u_i}{\partial y_i} = \mu > 0 \Leftrightarrow \frac{\lambda_i}{\mu} = \frac{1}{\frac{\partial u_i}{\partial y_i}}, \ \forall i.$$

Hence equation (x) may be written as

$$c'(x) = \sum_{i=1}^{n} \left(\frac{\lambda_i}{\mu} \right) \frac{\partial u_i}{\partial x} = \sum_{i=1}^{n} \frac{\frac{\partial u_i}{\partial x}}{\frac{\partial u_i}{\partial y_i}} = \sum_{i=1}^{n} RMS_i(x, y_i).$$

Therefore, a Pareto optimal allocation (x, y) is a feasible allocation, that is, such that

$$\sum_{i=1}^{n} y_i + c(x) = \sum_{i=1}^{n} \bar{y}_i,$$

satisfying

$$\sum_{i=1}^{n} RMS_i(x, y_i) = c'(x).$$

An Example: A Public Good

Example. Consider an economy in which each individual is endowed with 12 hours of time and cares exclusively about her consumption. There is a technology freely available that allows to produce K units of consumption good for each hour of labor used as input. The parameter K represents the *state of knowledge*, and is given by

$$K = \sum_{i=1}^{n} x_i,$$

where x_i is the number of hours individual i spends improving the technology.

Identify the Pareto optimal state of knowledge K^*.

An Example: Pareto Optimality

Maximizing the social surplus requires maximizing the economy's total output of consumption good (doesn't it?).

Letting $K \in [0, 12n]$ be total number of hours individuals spend improving the technolog and hence $12n - K$ the number of hours in production activities, we solve the problem

$$\max_{K \geq 0} K\left(12n - K\right).$$

(Of course, we need to be careful to allocate exactly 12 hours of production and technology improving activities to everyone.)

The solution to this problem is $K^* = 6n$. Thus, the optimal per-capita time allocated to improve the technology is 6 hours, and the per-capita consumption is $36n$.

An Exampe: Voluntary Contributions

Under voluntary contributions an individual decides the time she spends improving the technology by solving the problem

$$\max_{z \geq 0} (K_{-} + z)\,(12 - z)\,,$$

where K_{-} is the total number of hours the other individuals allocate to improving the technology.

The solution to this problem is

$$z^{*} = \frac{12 - K_{-}}{2}.$$

An Exampe: Voluntary Contributions

Let us assume that the (Nash) equilibrium of the (static, i.e., simultaneous) game individuals face is symmetric. (Actually, it is!) Then

$$z^* = \frac{12 - (n-1)z^*}{2} = \frac{12}{n+1}.$$

Thus, for $n > 1$ per-capita time allocated to improving the technology is $z^*(n) < 6$ and per-capita consumption is $144\left(n^2/(n+1)^2\right) < 36n$.

Voluntary contributions leads to under provision of the public good:

This is the Tragedy of the Commons!

Public Goods: Lindahl Equilibrium

Lindahl, observing the dual role of prices and quantities in markets, proposes a "solution" to the *free riding problem*.

The solution involves creating a "market" for public goods in which each individual pays a personalized price.

In the economy described above, a *Lindahl equilibrium* is a collection $(p^*, x^*, y^*) \in \mathbb{R}^n_+ \times \mathbb{R}_+ \times \mathbb{R}^n_+$ such that:

(1) $y_i^* = \bar{y}_i - p_i^* x^*$

(2) $c(x^*) = \sum_{i=1}^n p_i^* x^*$

(3) $x^* \in \arg\max u_i(x, \bar{y}_i - p_i^* x), \forall i \in \{1, ..., n\}$

(4) $\sum_{i=1}^n p_i^* = c'(x^*)$.

A Lindahl equilibrium allocation is Pareto optimal: By equations (1) and (2), it is feasible,

$$\sum_{i=1}^{n} y_i^* + c(x^*) = \sum_{i=1}^{n} \bar{y}_i - \sum_{i=1}^{n} p_i^* x + c(x^*) = \sum_{i=1}^{n} \bar{y}_i,$$

while by equations (3) and (4)

$$\sum_{i=1}^{n} RMS_i(x^*, y_i^*) = \sum_{i=1}^{n} p_i^* = c'(x^*).$$

Public Goods: Lindahl Equilibrium

Samuelson (1954) argues that while the Lindahl equilibrium is a useful concept (i.e., it identifies allocations satisfying desirable properties, such as

- Pareto optimality, and
- individual rationality,

the idea of setting a market for public goods is unworkable since each individualized market would be a monopsony.

The fundamental issue involved in *solving* the problem of public good provision is how to *elicit* (i.e., obtain) the information about individuals' preferences in order to design the system of personalized prices.

Public Goods: Mechanism Design

The issue raised by Samuelson (1954), that a fundamental part of the problem is that individuals' preferences are unknown, can be posed as a problem of *institution (or mechanism) design*. An earlier literature dealt with this issue framing the problem as a complete information game. (A very strong assumption!)

A mechanism is a pair (S, ϕ) given by

$$S = S_1 \times ... \times S_n,$$

where each S_i is a set of *actions or messages* individual $i \in \{1, ..., n\}$ can choose, and

$$\phi : S \to A$$

is an *outcome function* associating a feasible allocation $\phi(s) \in A \subset \mathbb{R}^n_+ \times \mathbb{R}_+$ to each profile of messages.

Walker (1973)'s mechanism *implements* the Lindahl allocation.

Public Goods: Walker's Mechanism

Consider a simple public good economy as describe above, in which the preferences of individual $i \in \{1, ..., n\}$, where $n > 2$, are represented by a utility function $u_i(x, y_i) = y_i + v_i(x)$, where $v_i : \mathbb{R}_+ \to \mathbb{R}$ is increasing and concave. Also, assume that the public good can be produced with constant returns to scale, i.e., $c(x) = \alpha x$, where $\alpha \geq 0$.

Walker's mechanism is given by (S, ϕ), where $S_1 = ... = S_n = \mathbb{R}$, and for $s \in \mathbb{R}^n$,

$$\begin{aligned} \phi_x(s) &= \sum_{j=1}^{n} s_j, \\ \phi_{y_i}(s) &= \bar{y}_i - p_i(s)\phi_x(s), \text{ where } p_i(s) = \frac{\alpha}{n} + (s_{i-1} - s_{i+1}). \end{aligned}$$

(For $i = 1$, we take $i - 1 := n$, and for $i = n$, we take $i + 1 := 1$.)

Example. Ann, Bob and Conrad share an apartment. The apartment has central heating and the temperature can be set at a cost $C(x) = cx$.

Calculate the equilibrium of Walker's mechanism assuming that their preferences for the temperature at the apartment (x) and income are represented by utility functions of the form

$$u_i(x, y_i) = \bar{y}_i - \alpha_i \left(t_i - x\right)^2,$$

where $(\alpha_A, t_A) = (3/2, 25)$, $(\alpha_B, t_B) = (1, 20)$, $(\alpha_C, t_C) = (1, 22)$, for the values of the constant marginal $c \in \{0, 2\}$.

Individual i's problem is:

$$\max_{s\in\mathbb{R}} \ \bar{y}_i - \left(\frac{c}{3} + (s_{i-1} - s_{i+1})\right)(s + s_{i-1} + s_{i+1}) \\ -\alpha_i \left(t_i - (s + s_{i-1} + s_{i+1})\right)^2 .$$

That is

$$-\left(\frac{c}{3} + (s_{i-1} - s_{i+1})\right) + 2\alpha_i(t_i - (s + s_{i-1} + s_{i+1})) = 0,$$

where $i-1 = C$ and $i+1 = B$ for $i = A$, $i-1 = A$ and $i+1 = C$ for $i = B$, and $i-1 = B$ and $i+1 = A$ for $i = C$.

Public Goods: Walker's Mechanism

Solving the system of FOCs for $i \in \{A, B, C\}$ we get

$$s^* = (s_A^*, s_B^*, s_C^*) = (\frac{131 - c}{21}, \frac{101 - 2c}{21}, \frac{245}{21})$$

Hence

$$x(s^*(c)) = \frac{159 - c}{7},$$

and

$$(p_A(s^*(c)), p_B(s^*(c)), p_C(s^*(c))) = \left(\frac{6c - 144}{21}, \frac{8c + 114}{21}, \frac{7c + 30}{21}\right).$$